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Momentum twistor was introduced by Andrew Hodges for the study of maximally supersymmetric theories. However, it is also very useful for simplifying spinor helicity expressions.

A momentum twistor is defined as

$$Z_i = \begin{pmatrix} \lambda_{i\alpha} \\ \mu_{i,\dot{\beta}} \end{pmatrix} \tag{1}$$

Holomorphic $\lambda_{i\alpha}$ and anti-holomorphic $\mu_{i,\dot{\beta}}$ are the basic objects in this formalism. The spinor helicity anti-holomorphic spinor is derived as,

$$\tilde{\lambda}_{i,\dot{\beta}} = \frac{\langle i, i+1\rangle \mu_{i-1,\dot{\beta}} + \langle i+1, i-1\rangle \mu_{i,\dot{\beta}} + \langle i-1, i\rangle \mu_{i+1,\dot{\beta}}}{\langle i, i+1\rangle\langle i-1, i\rangle}\,. \tag{2}$$

I. FOUR-POINT MOMENTUM TWISTOR

We use the choice

$$Z = \begin{pmatrix} 1 & 0 & -\frac{1}{x_1} & -\frac{1}{x_1} - \frac{1}{x_2} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Hence

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad \tilde{\lambda}_2 = \begin{pmatrix} -x_1 & 0 \end{pmatrix} \quad \tilde{\lambda}_3 = \begin{pmatrix} x_1 + x_2 & -x_2 \end{pmatrix} \quad \tilde{\lambda}_4 = \begin{pmatrix} -x_2 & x_2 \end{pmatrix} \tag{4}$$

Then $\langle 12\rangle = -1$, $[12] = x_1$, $\langle 14\rangle = -1$ and $[14] = x_2$. Hence we can replace

$$x_1 \to s, \quad x_2 \to t \tag{5}$$

Exercise For the four-point massless kinematics, try to convert

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} \tag{6}$$

to a function of Mandelstam variables.

Solution From the momentum twistor formula, $\langle 12\rangle = -1$, $\langle 34\rangle = -1/x_2$, $\langle 13\rangle = -1$ and $\langle 24\rangle = -1/x_1 - 1/x_2$. Then

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} = s/(s+t) \tag{7}$$

Now this computation is straightforward.

Claim For the four-point massless kinematics, any helicity-free and rational function of spinor product, is a rational function of x_1 and x_2, and hence a rational function of s and t.

II. FIVE-POINT MOMENTUM TWISTOR

In this case, the conversion between spinor helicity and momentum twistor variables are more complicated. On the other hand, since the five-point kinematics is intrinsically complicated, the five-point momentum twistor has a great advantage in real computations.

For the five-point massless kinematics, $p_i^2 = 0$, $i = 1, \ldots 5$, $p_1 + \ldots + p_5 = 0$, we usually use

$$s_{12}, \quad s_{23}, \quad s_{34}, \quad s_{45}, \quad s_{15} \tag{8}$$

as the independent Mandelstam variables. There is one Gram determinant

$$g(1,2,3,4) = \frac{1}{16}\Big(s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2s_{15}s_{23}s_{12}^2 - 2s_{23}^2 s_{34}s_{12} + 2s_{15}s_{23}s_{34}s_{12} - 2s_{15}^2 s_{45}s_{12}$$
$$+2s_{15}s_{23}s_{45}s_{12} + 2s_{15}s_{34}s_{45}s_{12} + 2s_{23}s_{34}s_{45}s_{12} + s_{23}^2 s_{34}^2 + s_{15}^2 s_{45}^2 + s_{34}^2 s_{45}^2 - 2s_{15}s_{34}s_{45}^2$$
$$-2s_{23}s_{34}^2 s_{45} + 2s_{15}s_{23}s_{34}s_{45}\Big) \tag{9}$$

Note that $\epsilon(1,2,3,4)^2 = -g(1,2,3,4)$. $\epsilon(1,2,3,4)$ is not a rational function of Mandelstam variables. For convenience, we define

$$\mathrm{tr}_5 \equiv 4i\epsilon(1,2,3,4) \tag{10}$$

We use the momentum-twistor choice,

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_2} & \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_4 & 1 \\ 0 & 0 & 1 & 1 & x_5/x_4 \end{pmatrix} \tag{11}$$

This was my choice in [1] and it was not the first choice nor the optimal choice. However, in [1] we first give the explicitly conversion formula.

From this choice, we have,

$$s_{12} = x_1, \quad s_{23} = x_2 x_4, \quad s_{34} = \frac{-x_1x_3 + x_1x_4x_3 + x_2x_4x_3 - x_2x_5x_3 + x_1x_2x_4}{x_2}$$
$$s_{45} = x_2\left(x_4 - x_5\right), \quad s_{15} = -x_3\left(x_5 - 1\right) \tag{12}$$

Note that the inverse map $(s_{12}, s_{23}, s_{34}, s_{45}, s_{15})$ to $(x_1, x_2, x_3, x_4, x_5)$ is not well defined since the above maps is not injective. For a generic value of s's, there are two sets of corresponding x's.

This problem can be resolved by the double covering trick. Instead of considering $(s_{12}, s_{23}, s_{34}, s_{45}, s_{15})$, we consider the "variety" $(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, \mathrm{tr}_5)$ with the constraint $\mathrm{tr}_5^2 = 16g(1,2,3,4)$. By this, we have the well-defined inverse map [1] ,

$$x_1 = s_{12} \tag{13}$$

$$x_2 = \frac{-s_{12}s_{15} + s_{12}s_{23} + s_{23}s_{34} + s_{15}s_{45} - s_{34}s_{45} - \mathrm{tr}_5}{2s_{34}} \tag{14}$$

$$\begin{aligned} x_3 = &-\frac{1}{2\left(s_{12} + s_{23} - s_{45}\right)s_{45}}(s_{23}\mathrm{tr}_5 - s_{45}\mathrm{tr}_5 + s_{12}s_{23}^2 - s_{34}s_{23}^2 - s_{12}s_{15}s_{23} \\ &- s_{12}s_{45}s_{23} - s_{15}s_{45}s_{23} + 2s_{34}s_{45}s_{23} + s_{15}s_{45}^2 - s_{34}s_{45}^2 - s_{12}s_{15}s_{45}) \end{aligned} \tag{15}$$

$$x_4 = \frac{s_{12}s_{15} - s_{12}s_{23} - s_{23}s_{34} - s_{15}s_{45} + s_{34}s_{45} - \mathrm{tr}_5}{2s_{12}\left(s_{15} - s_{23} + s_{45}\right)} \tag{16}$$

$$x_5 = \frac{\left(s_{23} - s_{45}\right)\left(-s_{12}s_{15} + s_{12}s_{23} + s_{23}s_{34} + s_{15}s_{45} - s_{34}s_{45} + \mathrm{tr}_5\right)}{2s_{12}s_{23}\left(-s_{15} + s_{23} - s_{45}\right)} \tag{17}$$

So we have a "birational" map between $(x_1, x_2, x_3, x_4, x_5)$ and $(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, \mathrm{tr}_5)$.

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[1] S. Badger, H. Frellesvig, Y. Zhang, A Two-Loop Five-Gluon Helicity Amplitude in QCD, JHEP 12 (2013) 045. `arXiv:1310.1051`, `doi:10.1007/JHEP12(2013)045`.